Reachability Problems for Continuous Linear Dynamical Systems

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Reachability for Continuous-Time Markov Chains

Bullmarket
Stagnant market
Bearmarket

Distribution $P(t)$ at time $t$ satisfies $P'(t) = P(t)Q$, where

$$Q = \begin{pmatrix} -0.025 & 0 & 0.005 \\ 0.3 & -0.5 & 0.2 \\ 0.02 & 0.4 & -0.42 \end{pmatrix}$$

is the rate matrix.

"Is it ever more likely to be a Bear market than a Bull market?"

$\exists t (P(t)_{Bear} \geq P(t)_{Bull})$
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- Reduce to the **time-bounded case** by computing the stationary distribution:

\[ \pi = (0.885, 0.071, 0.044) \]
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- Reduce to the **time-bounded case** by computing the stationary distribution:

  \[ \pi = (0.885, 0.071, 0.044) \]

- Require that \( \pi \) not be on boundary of the target set.
“To analyze a cyber-physical system, such as a pacemaker, we need to consider the discrete software controller interacting with the physical world, which is typically modelled by differential equations”

Rajeev Alur (CACM, 2013)
Hybrid automaton = states + variables $x \in \mathbb{R}^k$
Hybrid Automata: Various Continuous Dynamics

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  - $\dot{x} = 1$ $\Rightarrow$ timed automata
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  - $\dot{\mathbf{x}} = \mathbf{c}$ $\Rightarrow$ rectangular hybrid automata
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- ...

Is this location a trap?

$\dot{\mathbf{x}} = 3\mathbf{x} - \mathbf{y}$

$\dot{\mathbf{y}} = \mathbf{x} - 5\mathbf{y}$

$x := 2$

$y := 4$

$x \geq 10 \land y \leq 2$?
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$x \geq 10$
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Is this location a trap?

\[
\begin{align*}
\dot{x} &= 3x - y \\
\dot{y} &= x - 5y \\
x &= 2 \\
y &= 4
\end{align*}
\]

x ≥ 10 ∧ y ≤ 2?

Is ever more likely to be a Bear market than a Bull market:

\[\exists t \left( P(t)_{\text{Bear}} \geq P(t)_{\text{Bull}} \right) ?\]
\[
x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^k
\]
\[
\dot{x} = Ax
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Reachability for Continuous Linear Dynamical Systems

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\[ f(t) = \mathbf{u}^T \mathbf{x}(t) \]
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\[ f^{(k)}(t) + a_{k-1}f^{(k-1)}(t) + \ldots + a_1 f'(t) + a_0 f(t) = 0 \]
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Note – the \( \lambda_j \) are complex in general.
Let $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ be given as above, with all coefficients algebraic.
Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be given as above, with all coefficients algebraic.

**BOUNDDED-ZERO Problem**

*Instance:* $f$ and bounded interval $[a, b]$

*Question:* Is there $t \in [a, b]$ such that $f(t) = 0$?
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**Question**: Is there $t \in \mathbb{R}_{\geq 0}$ such that $f(t) = 0$?

- Decidability open! [Bell, Delvenne, Jungers, Blondel 2010]
A lot of work since 1920s on the zeros of exponential polynomials

\[ f(z) = \sum_{j=1}^{m} P_j(z)e^{\lambda_j z} \]

(Polya, Ritt, Tamarkin, Kac, Voorhoeve, van der Poorten, ...)
but mostly on distribution of complex zeros.
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CONTINUOUS-ORBIT Problem

The problem of whether the trajectory \( x(t) = e^{At}x(0) \) reaches a given target point was shown to be decidable by Hainry (2008) and in PTIME by Chen, Han and Yu (2015).
Our Results

Theorem (Chonev, Ouaknine, W. 2015)

Assuming Schanuel’s Conjecture, BOUNDED-ZERO is decidable at all orders.

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At order $\leq 8$, ZERO reduces to BOUNDED-ZERO.

Theorem (Chonev, Ouaknine, W. 2015)

At order 9, if ZERO is decidable then the Diophantine approximation type of any real algebraic number $\alpha$ is a computable number.

It turns out that decidability in the bounded case follows from a much more general result, discovered (but not published) in the early 1990s by Macintyre and Wilkie.

[Angus Macintyre, personal communication, July 2015]
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Schanuel’s Conjecture

**Theorem (Lindemann-Weierstrass)**

If $a_1, \ldots, a_n$ are algebraic numbers linearly independent over $\mathbb{Q}$, then $e^{a_1}, \ldots, e^{a_n}$ are algebraically independent.
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If \(a_1, \ldots, a_n\) are algebraic numbers linearly independent over \(\mathbb{Q}\), then \(e^{a_1}, \ldots, e^{a_n}\) are algebraically independent.

**Schanuel’s Conjecture**

If \(z_1, \ldots, z_n\) are complex numbers linearly independent over \(\mathbb{Q}\) then some \(n\)-element subset of \(\{z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}\}\) is algebraically independent.
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Easy Consequence

By Schanuel’s conjecture, some two-element subset of \( \{1, \pi i, e^1, e^{\pi i}\} \) is algebraically independent.
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If $z_1, \ldots, z_n$ are complex numbers linearly independent over $\mathbb{Q}$ then some $n$-element subset of $\{z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}\}$ is algebraically independent.

Theorem (Macintyre and Wilkie 1996)

The first-order theory of $(\mathbb{R}, +, \cdot, e^x)$ is decidable, assuming Schanuel’s conjecture.
The BOUNDED-ZERO Problem

Real-valued exponential polynomial $f(t) = \sum_{j=1}^{m} P_j(t)e^{\lambda_j t}$
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‘non-trivial’ zero \( \Rightarrow t^* \) transcendental
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Real-valued exponential polynomial \( f(t) = \sum_{j=1}^{m} P_j(t)e^{\lambda_j t} \)

Can this situation arise?
Real-valued exponential polynomial $f(t) = \sum_{j=1}^{m} P_j(t) e^{\lambda_j t}$

Easily! For example, $f(t) = 2 + e^{it} + e^{-it}$. 
Example

Write $f(t) = 2 + e^{it} + e^{-it}$ in the form $f(t) = P(e^{it})$ for the Laurent polynomial

$$P(z) = 2 + z + z^{-1}.$$
Laurent Polynomials and Factorisation

Example

- Write \( f(t) = 2 + e^{it} + e^{-it} \) in the form \( f(t) = P(e^{it}) \) for the Laurent polynomial

\[
P(z) = 2 + z + z^{-1}.
\]

- Factorisation \( P(z) = (1 + z)(1 + z^{-1}) \) induces a factorisation

\[
f(t) = \underbrace{(1 + e^{it})(1 - e^{it})}_{f_1(t)} \underbrace{f_2(t)}_{f_2(t)}
\]

Common zeros of \( f_1(t) \) and \( f_2(t) \) are tangential zeros of \( f(t) \).

Idea: factorise \( f(t) \). Noting that factors may be complex-valued!
Laurent Polynomials and Factorisation

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$$f(t) = (1 + e^{it}) (1 - e^{it})$$

$$f_1(t)$$

$$f_2(t)$$

Common zeros of $f_1$ and $f_2$ are tangential zeros of $f$. 
Example

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**Idea**: factorise $f$. 

### Example

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  - $f_1(t)$  
  - $f_2(t)$

- Common zeros of $f_1$ and $f_2$ are tangential zeros of $f$

**Idea:** factorise $f$. Noting that factors may be complex-valued!
Any exponential polynomial $f(t)$ can be written

$$f(t) = P(t, e^{a_1 t}, \ldots, e^{a_m t})$$

with

$$P \in \mathbb{C}[x, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$$

and $\{a_1, \ldots, a_m\}$ a set of complex algebraic numbers linearly independent over $\mathbb{Q}$. 
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**Proof Strategy**

Factorisation of $P$ into irreducible factors induces factorisation of $f$. Assuming Schanuel’s conjecture, we can decide the existence of zeros of real-valued and complex-valued irreducible factors.
ZERO Problem

Instance: $f$

Question: Is there $t \in \mathbb{R}_{\geq 0}$ such that $f(t) = 0$?
How well can one approximate a real number $x$ with rationals?

$$\left| x - \frac{p}{q} \right|$$
Diophantine Approximation

*How well can one approximate a real number $x$ with rationals?*

$$\left| x - \frac{p}{q} \right|$$

**Theorem (Dirichlet 1842)**

*There are infinitely many integers $p$, $q$ such that $\left| x - \frac{p}{q} \right| < \frac{1}{q^2}$.*
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**Theorem (Roth 1955)**

Let $x \in \mathbb{R}$ be algebraic. Then for any $\varepsilon > 0$ there are finitely many integers $p$, $q$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\varepsilon}}.$$
Diophantine Approximation

How well can one approximate a real number $x$ with rationals?

$$|x - \frac{p}{q}|$$

Theorem (Dirichlet 1842)

There are infinitely many integers $p, q$ such that $|x - \frac{p}{q}| < \frac{1}{q^2}$.

Definition

Let $x \in \mathbb{R}$. The Diophantine-approximation type $L(x)$ is:

$$L(x) = \inf \left\{ c : |x - \frac{p}{q}| < \frac{c}{q^2} \text{ for some } p, q \in \mathbb{Z} \right\}.$$
Finite continued fractions:

\[
[3, 7, 15, 1, 292] = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292}}}}
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\[ = 3.141592653 \ldots \]
Continued Fractions

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= 3.141592653 \ldots

Infinite continued fractions:

\[ [a_0, a_1, a_2, a_3, \ldots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}} \]
Theorem

The continued fraction expansion of a real quadratic irrational number is periodic.
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\[ \sqrt{389} = [19, 1, 2, 1, 1, 1, 1, 2, 1, 38, 1, 2, 1, 1, 1, 2, 1, 38, \ldots] \]
Theorem

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\[ \sqrt{389} = [19, 1, 2, 1, 1, 1, 1, 2, 1, 38, 1, 2, 1, 1, 1, 1, 2, 1, 38, \ldots ] \]

What about numbers of degree \( \geq 3 \)?
Theorem

The continued fraction expansion of a real quadratic irrational number is periodic.

\[ \sqrt{389} = [19, 1, 2, 1, 1, 1, 1, 2, 1, 38, 1, 2, 1, 1, 1, 1, 2, 1, 38, \ldots] \]

What about numbers of degree \( \geq 3 \)?

\[ \sqrt[3]{2} = [1, 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, 2, 1 \]
\[ \quad \quad 3, 4, 1, 1, 2, 14, 3, 12, 1, 15, 3, 1, 4, 534, 1, 1, 5, 1, 1, \ldots] \]
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What about numbers of degree \( \geq 3 \)?

\[ 3\sqrt{2} = [1, 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, 2, 1, 3, 4, 1, 1, 2, 14, 3, 12, 1, 15, 3, 1, 4, 534, 1, 1, 5, 1, 1, \ldots] \]

Lang and Trotter: “no significant departure from random behaviour”
“[…] no continued fraction development of an algebraic number of higher degree than the second is known. It is not even known if such a development has bounded elements.”

“[...] no continued fraction development of an algebraic number of higher degree than the second is known. It is not even known if such a development has bounded elements.”


“Is there an algebraic number of degree higher than two whose simple continued fraction has unbounded partial quotients? Does every such number have unbounded partial quotients?”

R. K. Guy, 2004
Fact. The simple continued fraction expansion of $x \in \mathbb{R}$ is unbounded iff $L(x) = 0$. 
**Fact.** The simple continued fraction expansion of \( x \in \mathbb{R} \) is unbounded iff \( L(x) = 0 \).

**Theorem (Chonev, Ouaknine, W., 2015)**

If the ZERO PROBLEM is decidable at order 9 then there is an algorithm that given a real algebraic number \( \alpha \) computes \( L(\alpha) \) to arbitrary precision. In particular, the set

\[
\{ \alpha \in \overline{\mathbb{Q}} : \alpha \text{ has bounded partial quotients} \}
\]

would be recursively enumerable.
The ZERO Problem at Low Orders

ZERO Problem

Instance: $f$

Question: Is there $t \in \mathbb{R}_{\geq 0}$ such that $f(t) = 0$?
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Theorem

*In dimension 8 or less, ZERO reduces to BOUNDED-ZERO.*
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In the limit $f$ is either never zero or infinitely often zero, and we can decide which is the case.
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### Theorem

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Diophantine-approximation bounds play a key role in the proof—specifically Baker’s theorem on linear forms in logarithms of algebraic numbers.
Consider the exponential polynomial

$$f(t) = 2 + \cos(t + \varphi_1) + \cos(\sqrt{2}t + \varphi_2) - e^{-t}$$
Illustrative Example

Consider the exponential polynomial

\[ f(t) = 2 + \cos(t + \varphi_1) + \cos(\sqrt{2}t + \varphi_2) - e^{-t} \]

Orbit \( \{(t + \varphi_1, \sqrt{2}t + \varphi_2) \mod 2\pi : t \in \mathbb{R}_{\geq 0}\} \) is dense in \([0, 2\pi]^2\)
Consider the exponential polynomial

\[ f(t) = 2 + \cos(t + \varphi_1) + \cos(\sqrt{2}t + \varphi_2) - e^{-t} \]

Orbit \{ (t + \varphi_1, \sqrt{2}t + \varphi_2) \mod 2\pi : t \in \mathbb{R}_{\geq 0} \} is dense in \([0, 2\pi]^2\)
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Baker’s Theorem:

\[ \left\| \left( t + \varphi_1, \sqrt{2}t + \varphi_2 \right) - (\pi, \pi) \right\| \geq \frac{1}{\text{poly}(t)} \]
Conclusion and Perspectives
A **linear recurrence sequence** is a sequence \( \langle u_0, u_1, u_2, \ldots \rangle \) of integers such that there exist constants \( a_1, \ldots, a_k \), such that

\[
 u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \ldots + a_k u_n
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for all \( n \geq 0 \).
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**Theorem (Skolem 1934; Mahler 1935, 1956; Lech 1953)**

The set of zeros of a linear recurrence sequence is semi-linear:

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    \{ n : u_n = 0 \} = F \cup A_1 \cup \ldots \cup A_\ell
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where \( F \) is finite and each \( A_i \) is a full arithmetic progression.
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**Theorem (Berstel and Mignotte 1976)**

In Skolem-Mahler-Lech, the infinite part (arithmetic progressions $A_1, \ldots, A_\ell$) is fully constructive.
The Skolem Problem

Skolem Problem
Does $\exists n$ such that $u_n = 0$ ?

“It is faintly outrageous that this problem is still open; it is saying that we do not know how to decide the Halting Problem even for ‘linear’ automata!”

“... a mathematical embarrassment ...”

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Continuous Skolem Problem

Does $\exists t$ such that $f(t) = 0$?
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Does \( \exists t \) such that \( f(t) = 0 \)?

- Even the bounded problem is hard.
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Similar obstacles for the Infinite-Zeros Problem.
Wrapping Things Up

Continuous Skolem Problem

Does $\exists t$ such that $f(t) = 0$?

- Even the bounded problem is hard.
- Formidable “mathematical obstacle” at dimension 9 in the unbounded case.
- Similar obstacles for the Infinite-Zeros Problem.
- Diophantine-approximation techniques unavoidable.